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A Cauchy-Schwarz inequality for expectation of matrices

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Abstract

A generalization of the Cauchy-Schwarz inequality for expectations of matrices is proved.

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Notation. For any $n \times p$ matrix A, A' denotes its transpose and $\|A\|$ its Euclidean norm, that is $\|A\| = \sqrt{\sum_{1 \leq i \leq n, 1 \leq j \leq p} a_{ij}^2}$.

Tripathi (1999) proved the following extension of the Cauchy-Schwarz inequality.

Lemma 1 *Let $x \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$ be random vectors such that $\mathbb{E}\|x\|^2 < \infty$, $\mathbb{E}\|y\|^2 < \infty$, and $\mathbb{E}yy'$ is non-singular.*

Then $\mathbb{E}\left(xx'\right) - \mathbb{E}\left(xy'\right)\mathbb{E}^{-1}\left(yy'\right)\mathbb{E}\left(yx'\right)$ is positive semidefinite.

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Though this inequality looks astonishingly familiar, Tripathi noted that he had been unable to discover any reference in the literature. My own literature review has been similarly unfruitful. Such an inequality is however useful for studying the asymptotic variance of econometric estimators. Tripathi provides an example based on the estimator proposed by Robinson (1988) in the partial linear model. For easy applications to wider contexts, one needs a generalization to random matrices, which does not follow from the above result. For instance, Powell (1994) reviews asymptotic efficiency bound of a class of semiparametric estimators and relies on " a simple extension of the Gauss-Markov argument," which in turn relies on some form of the Cauchy-Schwarz inequality.

I here provide a matrix extension of the Cauchy-Schwarz inequality for expectations, as well as a simpler and more straightforward proof than Tripathi (1999).

Lemma 2 *Let* $A \in \mathbb{R}^{n\times p}$ *and* $B \in \mathbb{R}^{n\times q}$ *be random matrices such that* $\mathbb{E}\|A\|^2 < \infty$, $\mathbb{E}\|B\|^2 < \infty$, *and* $\mathbb{E}\left(A'A\right)$ *is non-singular.*

Then $\mathbb{E}\left(B'B\right) - \mathbb{E}\left(B'A\right)\mathbb{E}^{-1}\left(A'A\right)\mathbb{E}\left(A'B\right)$ *is positive semidefinite, with equality iff* $B = A\mathbb{E}^{-1}\left(A'A\right)\mathbb{E}\left(A'B\right)$.

Proof. Consider $\Lambda = \mathbb{E}^{-1}\left(A'A\right)\mathbb{E}\left(A'B\right) \in \mathbb{R}^{p\times q}$. Then

$$\mathbb{E}\left[\left(B - A\Lambda\right)'\left(B - A\Lambda\right)\right] = \mathbb{E}\left(B'B\right) - \mathbb{E}\left(B'A\right)\mathbb{E}^{-1}\left(A'A\right)\mathbb{E}\left(A'B\right)$$

is positive semidefinite by definition, as the expectation of a matrix product of the form $C'C$, and is zero if and only if $B = A\Lambda$. ∎

When A and B are vectors of equal dimension, Lemma 2 reduces to Lemma 1. Note also that a straightforward corollary of Lemma 2 is as follows.

Corollary 1 *Let* $A \in \mathbb{R}^{n\times 1}$ *and* $B \in \mathbb{R}^{n\times 1}$ *be random vectors such that* $\mathbb{E}\|A\|^2 < \infty$, $\mathbb{E}\|B\|^2 < \infty$, *and* $\mathrm{Var}\left(A\right)$ *is non-singular.*

Then $\mathrm{Var}\left(B\right) - \mathrm{Cov}\left(B, A\right)\mathrm{Var}^{-1}\left(A\right)\mathrm{Cov}\left(A, B\right)$ *is positive semidefinite, with equality iff* $B - \mathbb{E}(B) = \left(A - \mathbb{E}(A)\right)\mathrm{Var}^{-1}\left(A\right)\mathrm{Cov}\left(A, B\right)$.

To prove this corollary, replace the vectors A and B by $A-\mathbb{E}(A)$ and $B-\mathbb{E}(B)$ in Lemma 2. This result is known, see e.g. Property B.20 in Gourieroux and Monfort (1989), but the usual method of proof is different and not as direct.

As an application of Lemma 2, consider a model defined by conditional moment restrictions

$$\mathbb{E}\left[\rho(Z,\theta)|X\right]=0$$

and the GMM estimator based on the conformable matrix of instruments $W(\cdot)$ defined as

$$\widehat{\theta}=\arg\min\widehat{g}_n(\theta)P\widehat{g}_n(\theta), \qquad \widehat{g}_n(\theta)=n^{-1}\sum_{i=1}^{n}W(X_i)\rho(Z_i,\theta)\,.$$

Let us define $\nabla_\theta\rho(Z,\theta)=\frac{\partial\rho(Z,\theta)}{\partial\theta}$, $V=\mathbb{E}\left[W(X)\rho(Z,\theta)\rho'(Z,\theta)W'(X)\right]$, and $G=\mathbb{E}\left[W(X)\nabla'_\theta\rho(Z,\theta)\right]$. The inverse of the asymptotic $\sqrt{n}$-variance of $\widehat{\theta}$ is then $G'PG\left(G'PVPG\right)^{-1}G'PG$, see for instance Hansen (1982). A first application of Lemma 2 with $A=V^{1/2}PG$ and $B=V^{-1/2}G$ yields as an upper bound $G'V^{-1}G$, which is attained for $P=V^{-1}$. (As neither A nor B are random, the result reduces to a known one.) A second application with $A=\mathrm{Var}^{1/2}\left(\rho(Z,\theta)|X\right)W'(X)$ and $B=\mathrm{Var}^{-1/2}\left(\rho(Z,\theta)|X\right)\nabla'_\theta\rho(Z,\theta)$ yields

$$\mathbb{E}\left(B'B\right)=\mathbb{E}\left[\nabla_\theta\rho(Z,\theta)\mathrm{Var}^{-1}\left(\rho(Z,\theta)|X\right)\nabla'_\theta\rho(Z,\theta)\right]$$

as an upper bound in $W(X)$. This is the inverse of semiparametric asymptotic efficiency bound derived by Chamberlain (1987). It is straightforward to see that the criterion for equality is satisfied when $W(X)$ equals $\nabla_\theta\rho(Z,\theta)\mathrm{Var}^{-1}\left(\rho(Z,\theta)|X\right)$.

Note. I am using on purpose "Cauchy-Schwarz inequality for expectation of matrices" to make a difference with the mathematical literature focusing on extensions of the Cauchy-Schwarz inequality for non-random matrices.

REFERENCES

Chamberlain, G. (1987). Asymptotic efficiency in estimation with conditional moment restrictions. *J. Econometrics* **34(3)**, 305-334.

Gourieroux, C., and Monfort, A. (1989). *Statistics and Econometric Models*. Cambridge University Press: Cambridge.

Hansen, L.P. (1982). Large sample properties of Generalized-Method of Moments estimators. *Econometrica* **50(4)**, 1029-1054.

Powell, J.L. (1994). Estimation of semiparametric models. Chapter 41, 2443-2521, in *Handbook of Econometrics*, Engle, R., and McFadden, D., eds. North-Holland: Amsterdam.

Robinson, P. (1988). Root-N consistent semiparametric regression. *Econometrica* **56(4)**, 931-954.

Tripathi, G. (1999). A matrix extension of the Cauchy-Schwarz inequality. *Economics Letters* **63(1)**, 1-3.